Recent developments in production, cost, and index number theory, with an application to international differences in the cost and efficiency of steelmaking in 1907/9

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I Introduction

In the middle of the nineteenth century the British iron and steel industry was the largest in the world and its exports dominated international markets. By the First World War, the American and German industries produced considerably more steel than the British and were major exporters. Britain, indeed, had become the world's largest importer of iron and steel. The immediate cause of this reversal (at least the reversal in international trade) was a change in relative production costs: in the middle of the nineteenth century British costs were lower than German or American costs, but by 1913 the latter two industries produced more cheaply than Britain. This paper is concerned with understanding why Germany and America produced steel less expensively than Britain in the first decade of the twentieth century.

In this paper it will be assumed that steel production exhibits constant returns to scale so that long run average total cost is independent of the rate of production. In that case, it is intuitively clear that differences in the prices of steelmaking inputs and differences in the efficiency of production are the two factors that might account for differences in unit costs. To explain the differences in international steelmaking costs in the early twentieth century, therefore, one must ascertain the relative importance of efficiency differences and input price differences. (After this task is completed, the analysis can go on to explain these differences themselves.) Recent work in duality theory and the theory of index numbers provides the basis for this decomposition. Since the problem is so common in economic history, we shall consider it thoroughly both from a theoretical and a practical point of view. Then the theory will be applied to the problem of ascertaining and decomposing relative production costs in Britain, Germany and America at the time of their industrial censuses of 1907 and 1909.

II Productivity measurement and Cost Decomposition

There is no point developing theory independently of the data it will be applied to, so we begin by specifying the data we intend to analyze. The data pertain to two firms or industries (values for which are denoted by superscripts 0 and 1). The two industries might be contemporaneous (i.e. the British and German steel industries in 1907) or they might be the same industry or firm at two times (i.e. the German steel industry in 1870 and 1910). For each industry, the investigator observes output, Q^0 and Q^1, the vectors of the quantities of the N inputs consumed, $Z^0=(Z_1^0, \ldots, Z_n^0)$ and $Z=(Z_1^1, \ldots, Z_n^1)$, and the vectors of prices of those inputs $w^0=(w^0, \ldots, w_n^0)$ and $w^1=(w^1, \ldots, w_n^1)$. For instance, Q might be steel production in a year, the elements of Z might be total man-hours worked, tons of iron ore smelted, tons of coke consumed, etc., in the same year, and the corresponding elements of w would be the wage rate per hour and the price per ton of ore and coke. Clearly, one can divide the total consumption of an input by the corresponding output rate, to determine unit input consumption:

$x^0=Z^0/Q^0=(Z_1^0/Q^0, \ldots, Z_n^0/Q^0)=(x_1^0, \ldots, x_n^0)$ and

$x^1=Z^1/Q^1=(Z_1^1/Q^1, \ldots, Z_n^1/Q^1)=(x_1^1, \ldots, x_n^1)$. The data are specified in this way since these are the sorts of data one might hope to obtain form two industrial censuses or from the income statements of two firms.

One can directly compute unit production costs for the two industries, $w^0 \cdot x^0=\sum_{i=1}^{n} w_i^0\, x_i^0$ and $w^1 \cdot x^1=\sum_{i=1}^{n} w_i^1\, x_i^1$, and form their ratio $w^1 \cdot x^1/w^0 \cdot x^0$. This number is relative production costs in the two cases. *Our object is to work out how to express $w^1 \cdot x^1/w^0 \cdot x^0$ as the product of two terms, one of which captures the effect on costs of any differences in efficiency that might obtain between the two industries, and the other of which encompasses the effect on costs of any differences in the prices the two industries (or firms) pay for their inputs.* Only by computing these two terms can we talk clearly about the effect of efficiency differences and input price differences on relative production costs.

It is simplest to start by considering the problem of measuring efficiency differences. Economists usually define greater efficiency to be the "capacity to produce more output from a given bundle of inputs" and that is the pertinent concept for the problem at hand. We assume that the technologies of the two firms can be represented by production functions and that the functions are identical up to a multiplicative coefficient: $Q^0=A^0 \cdot f(Z^0)$ and $Q^1=A^1 \cdot f(Z^1)$. f is assumed to be a linearly homogeneous neoclassical production function. Since Q increases with A for an unchanging Z, A indexes efficiency in the sense we are using it here. The problem of measuring efficiency differences is, therefore, the problem of determining the relative differences in A, i.e. ascertaining A^1/A^0, from the quantities and prices of the inputs and outputs in the two situations. If f were known, A^1/A^0 could be imputed by direct substitution:

$$\frac{A^1}{A^0}=\frac{Q^1/Q^0}{f(Z^1)/f(Z^0)} \tag{1}$$

In general, we do not know f so this straightforward calculation is not feasible. Later, we shall see how different input quantity indices might be used to estimate $f(Z^1)/f(Z^0)$. At the moment, however, one might notice that the numerator of equation 1 is relative output and the denominator is a ratio (mediated by f) of relative inputs, so the equation is a ratio of "total output" to "total input". We shall refer to $f(Z^1)/f(Z^0)$ as the "true input quantity index" and to A^1/A^0 as the "true total factor productivity index".

To decompose relative unit costs into efficiency and input price terms, one must introduce further assumptions about the input markets and the behaviour of the firms or industries. We shall assume that Z^0 and Z^1 are available in perfectly elastic supply at prices w^0 and w^1. Further, we shall assume the industries minimize production costs given those input prices and their production functions. Minimized total costs then depend on total production, input prices, the level of efficiency, and the form of the function f. Since we are assuming constant returns to scale, we can speak equally well of unit costs, which depend only on input prices, efficiency and f. Moreover, since the efficiency term A was assumed to be multiplicatively separable in the production function, the unit cost function, which shows how unit costs depend on efficiency and input prices, has a particular form:

$$w^0 \cdot x^0 = \frac{c(w^0)}{A^0} \tag{2a}$$

$$w^1 \cdot x^1 = \frac{c(w^1)}{A^1} \tag{2b}$$

The functional form of c is determined by – in the jargon is "dual to" – f. A represents the impact of efficiency on unit costs. Since A is in the denominator, increases in A lower costs. c(w) represents the inpact of input princes on costs. It can be shown that c is increasing in w and linearly homogeneous as well, so that increases in input prices raise costs and equiproportional increases in the prices of all inputs raise costs in the same proportion.

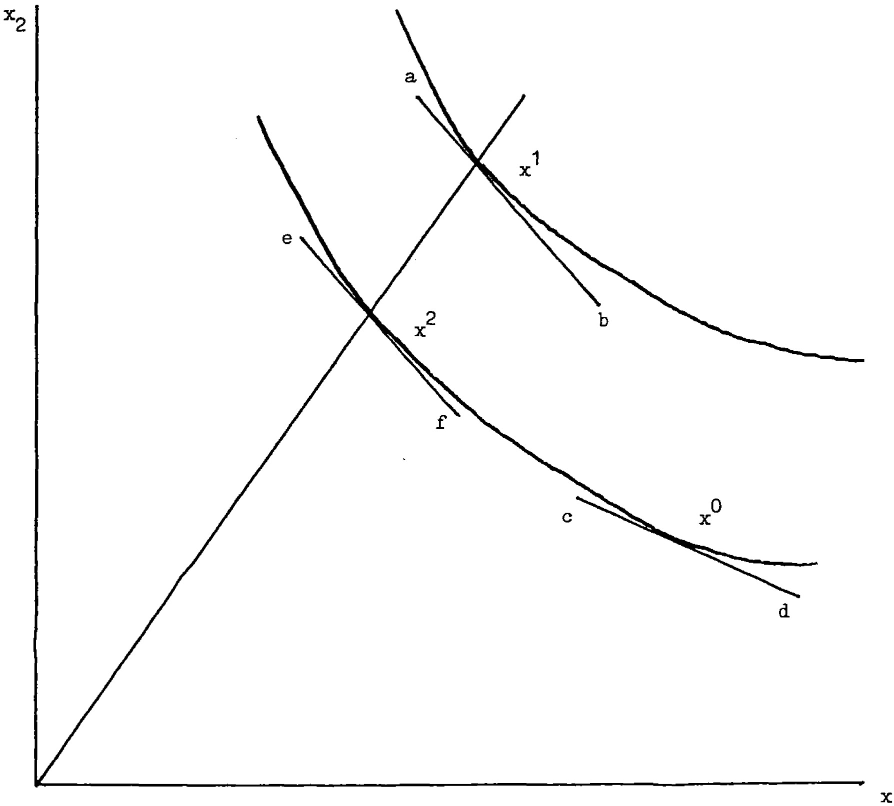
By dividing equation 2b by equation 2a, we obtain an equation for decomposing unit costs into efficiency and input price effects:

$$\frac{w^1 \cdot x^1}{w^0 \cdot x^0} = \frac{A^0}{A^1} \cdot \frac{c(w^1)}{c(w^0)} \tag{3}$$

The left side – relative unit costs – is observable. A^0/A^1 is the term that indicates the contribution of the difference in efficiency to the difference in costs. $c(w^1)/c(w^0)$, which is called the "true input price index", represents the effect of input price differences on costs.

One can imagine proceeding in either of two ways. If c were known, $c(w^1)/c(w^0)$ could be computed directly, and then A^0/A^1 could be calculated by deflating relative unit costs (the left side of equation 3) by the true input price index. Comparing A^0/A^1 and $c(w^1)/c(w^0)$ would then show the relative contributions of efficiency and input price differences on unit cost differences. Unfortunately, c is not known in general, but we shall shortly show how to approximate the true input index by computable price indices that allow the practical application of this procedure. Alternatively, of course, one could compute A^0/A^1 from equation 1 and proceed in a parallel manner to the same end. Analagous index number problems still arise, however, as we have already noted.

Before considering the solution of these index number problems, we can give the theory a geometric interpretation in terms of the standard isoquant diagram. Since we are assuming constant returns to scale, we can simplify the geometry by working only with unit isoquants. Figure 1 shows these isoquants for the case of two inputs x_1 and x_2. The points $x^0=(x_1^0, x_2^0)$ and $x^1=(x_1^1, x_2^1)$ are the observed unit input vectors for the two industries, and the unit isoquants are drawn through them. Since the pro-



Note: ab and ef are parallel as are the two unit isoquants.

Figure 1: The Geometry of Cost Decomposition

duction functions are identical up to the multiplicative efficiency term A, the isoquants are parallel, i.e. the isoquants have equal slopes for points that intersect the same ray from the origin. It is assumed that x^0 and x^1 are cost minimizing input combinations so the slopes of the tangents to the isoquants at the points (lines ab and cd) equal the prevailing input price ratios. x^0 and x^1 and the slopes of ab and cd are observable. Point x^2 is not observable. x^2 is the input combination on the isoquant through x^0 that would minimize costs at the input prices w^1. (line segment ef is parallel to ab.) Since the isoquants are parallel, x^2 is on the same ray from the origin as x^1.

The following identity is obviously true:

$$\frac{w^1 \cdot x^1}{w^0 \cdot x^0} = \frac{w^1 \cdot x^1}{w^1 \cdot x^2} \cdot \frac{w^1 \cdot x^2}{w^0 \cdot x^0} \tag{4}$$

$w^0 \cdot x^0$ is the unit cost of production for firm 0 at input prices w^0, and $w^1 \cdot x^2$ is the unit production cost of the same firm at prices w^1. Hence, by equation 2a, $w^0 \cdot x^0 = c(w^0)/A^0$ and $w^1 \cdot x^2 = c(w^1)/A^0$.

What of the term $w^1 \cdot x^1 / w^1 \cdot x^2$? Since x^1 and x^2 are on the same ray through the origin, $x^2 = \lambda x^1$ where λ is a scalar. Substituting $x^2 = \lambda x^1$ into $w^1 \cdot x^1 / w^1 \cdot x^2$ yields $w^1 \cdot x^1 / w^1 \cdot \lambda x^1 = 1/\lambda$. But what is the meaning of λ? It equals the true total factor productivity index. To see that, recall that x^1 is on the unit isoquant for industry 1 so $1 = A^1 \cdot f(x^1)$. x^2 is, likewise, on the unit isoquant for industry 0; hence, $1 = A^0 \cdot f(x^2)$. Equating these expressions and substituting $x^2 = \lambda x^1$ yields:

$$\begin{aligned} A^1 \cdot f(x^1) &= A^0 \cdot f(x^2) \\ &= A^0 \cdot f(x^1) \\ &= \lambda A^0 \cdot f(x^1) \end{aligned}$$

since f is linearly homogeneous. Division gives the desired result:

$$\frac{A^1}{A^0} = \lambda.$$

Making the substitutions $w^0 \cdot x^0 = c(w^0)/A^0$, $w^1 \cdot x^2 = c(w^1)/A^0$, and $w^1 \cdot x^1 / w^1 \cdot x^2 = A^0/A^1$, equation 4 becomes

$$\frac{w^1 \cdot x^1}{w^0 \cdot x^0} = \frac{A^0}{A^1} \cdot \frac{c(w^1)}{c(w^0)}$$

which is equation 3.

We can now interpret the terms of equation 3 in terms of the geometry of Figure 1. Relative unit costs equals the product of two terms. The first term A^0/A^1, is the efficiency difference or the relative distance the two isoquants are from the origin. The second term, $c(w^1)/c(w^0)$, equals the impact on costs as one "slides along" an isoquant (i.e. adjusts the cost minimizing input mix) in response to differences in input prices between industries 0 and 1.

To apply equation 3, i.e. to decompose relative units costs into efficiency and input price terms, one must either ascertain $f(Z^1)/f(Z^0)$ in equation 1 or $c(w^1)/c(w^0)$ in equation 3 or both. In practice, one uses input quantity and input price indices to approximate these "true" indexes. There is a vast - indeed an infinite - number of indices one might use. Which should be chosen? Considerable progress has been made by economists in recent years in solving this problem. A fundamental notion in this work is that of "exactness". An input quantity index, for instance, is exact for a particular production function f(Z), if the index number equals $f(Z^1)/f(Z^0)$. Similarly an input price index is exact for a unit cost function c(w) if the index equals $c(w^1)/c(w^0)$. Perhaps the most obvious exactness relationship is that a geometric input index is exact for a Cobb-Douglas production function. Mathematical economists have worked out the functions for which common index numbers are exact, and vice versa. Some of these results are summarized in Table 1. The results are stated in terms of production functions and input quantity indices but analogous results are true for cost functions and input price indices. Notice that Paasche and Laspeyres indices are both exact for both Leontief and linear functions. Exactness relations are not unique. Exactness results have also been derived for a more general function that includes the Törnqvist and square-root-quadratic functions as special cases. There are an infinite

Table 1: Exactness Relationships

Production Function		Corresponding Exact Index Number	
Name	Equation	Name	Equation
Leontief	$f(Z) = \mathrm{Min}\left\{\frac{Z_1}{b_1}, \ldots, \frac{Z_n}{b_n}\right\}$	Laspeyres	$\frac{f(Z^1)}{f(Z^0)} = \frac{\sum_{i=1}^{n} w_i^0 Z_i^1}{\sum_{i=1}^{n} w_i^0 Z_i^0}$
linear	$f(Z) = \sum_{i=1}^{n} a_i Z_i$	Paasche	$\frac{f(Z^1)}{f(Z^0)} = \frac{\sum_{i=1}^{n} w_i^1 Z_i^1}{\sum_{i=1}^{n} w_i^1 Z_i^0}$
Cobb-Douglas	$\ln f(Z) = \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln Z_i$ where $\sum_{i=1}^{n} \alpha_i = 1$	geometric	$\frac{f(Z^1)}{f(Z^0)} = \prod_{i=1}^{n} \left[\frac{Z_i^1}{Z_i^0}\right]^{s_i}$
translog	$\ln f(Z) = {}_0 + \sum_{i=1}^{n} \alpha_i \ln Z_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \ln Z_i \ln Z_j$ where $\sum_{i=1}^{n} \alpha_i = 1,\ \alpha_{ij} = \alpha_{ji}$ for all i,j and $\sum_{i=1}^{n} \alpha_{ij} = 0$ for $i = 1, \ldots, N$	Törnqvist	$\frac{f(Z^1)}{f(Z^0)} = \prod_{i=1}^{n} \left[\frac{Z_i^1}{Z_i^0}\right]^{(s_i^0 + s_i^1)/2}$
square-root-quadratic	$f(Z) = \left[\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} Z_i Z_j\right]^{\frac{1}{2}}$ where $a_{ij} = a_{ji}$ for all ij	Fisher ideal	$\frac{f(Z^1)}{f(Z^0)} = \left[\frac{\sum_{i=1}^{n} w_i^0 Z_i^1}{\sum_{i=1}^{n} w_i^0 Z_i^0}\right]^{\frac{1}{2}} \left[\frac{\sum_{i=1}^{n} w_i^1 Z_i^1}{\sum_{i=1}^{n} w_i^1 Z_i^0}\right]^{\frac{1}{2}}$

number of production functions and corresponding exact index numbers to choose from. It must also be emphasized that these exactness relations only obtain if the firms or industries concerned have minimized costs. In Table 1, the symbols s_i, s_i^0, and s_i^1 refer to the shares in cost of input i.

In his fundamental paper, "Exact and Superlative Index Numbers", Diewert[1] has suggested that one can discriminate among index numbers on the basis of the production and cost functions for which they are exact. Some functions (e.g. Cobb-Douglas and Leontief functions) can be shown to be first-order approximations to any constant returns-to-scale production function whereas other functions (e.g. translog and square-root-quadratic) can be shown to be second-order approximations to such production functions. Since second-order functions would be expected to fit the data better, Diewert urges that index numbers exact for such functions ought to be preferred to index numbers exact for first-order functions. Diewert calls the index numbers that are exact for second-order approximating functions "superlative" index numbers.

1. Diewert, W. E., *Exact and Superlative Index Numbers,* in: Journal of Econometrics, 4 (1976), pp. 115–145.

The Törnqvist and Fisher idea index numbers shown in Table 1 are superlative. Diewert has found that the dispersion among superlative index numbers is generally less than the dispersion among indexes exact for first order approximators when all are evaluated for the same set of data.

In a more recent paper, Allen and Diewert[2] have proposed another criterion for solving the index number problem. The object of the index number, of course, is to ascertain $f(Z^1)/f(Z^0)$ and $c(w^1)/c(w^0)$. Since f and c are both linearly homogeneous, it can be shown that they are bounded by Paasche and Laspeyres indices, irrespective of the functional form of f or c. In other words, Paasche and Laspeyres input price indices bound $c(w^1)/c(w^0)$, and Paasche and Laspeyres input quantity indices bound $f(Z^1)/f(Z^0)$ so long as f exhibits constant returns to scale. This result is convenient if the Paasche and Laspeyres indices are close together, for then one may closely bound the cost decomposition without worrying further about the choice of an index number. Provided either the input prices, w^1 and w^0, or the input quantities, Z^1 and Z^0, be roughly proportional, the bounds will be tight and the problem of choosing an index number satisfactorily finessed.

One is tempted to go somewhat further. The Fisher ideal index is a superlative index number and so favoured by Diewert's original criterion. Further, since it is the geometric mean of the Paasche and Laspeyres indices, it always lies within those bounds. No other superlative index number has this property. Unless one had extraveous information as the form of f or c, the Fisher ideal index might always be preferred since it always satisfies both criteria.

III Productivity and Steelmaking Costs, 1907/9

We shall now apply the theory developed in the last section to the problems of measuring productivity, input prices, and costs in the British, German and American steel industries in the early twentieth century.[3]

Equation 3 will be the fundamental tool. In the last section, it was suggested that either A^0/A^1 or $c(w^1)/c(w^0)$ could be determined residually by dividing $w^1 \cdot x^1/w^0 \cdot x^0$ by the other. In this section, we will use the equation differently. A^0/A^1 and $c(w^1)/c(w^0)$ will be estimated directly and $w^1 \cdot x^1/w^0 \cdot x^0$ computed as their product.

First, the difference in total factor productivity (A^0/A^1) among the three countries must be determined. Equation 1 is the relevant equation for this task. f will be assumed to be Cobb-Douglas so a geometric index of inputs will be used to compute $f(Z^1)/f(Z^0)$. In that case,

$$\frac{A^1}{A^0} = \frac{Q^1/Q^0}{\prod_{i=1}^{n} \left[\frac{x_i^1}{x_i^0}\right]^{s_i}} = \prod_{i=1}^{n} \left[\frac{Q^1/x_i^1}{Q^0/x_i^0}\right]^{s_i} \tag{5}$$

2. Allen, R. C., and Diewert, W. E., *Direct Versus Implicit Superlative Index Number Formulae,* in: Review of Economics and Statistics, 63 (1981).

3. The numbers discussed in this section were originally published in Allen, R. C., *International Competition in Iron and Steel, 1850–1913,* in: Journal of Economic History, 39 (1979), pp. 911–937. Readers are referred to that paper for sources and elaboration.

The right hand equality follows since the shares sum to 1. The difference in efficiency (total factor productivity) is a weighted geometric average of the relative average products of the inputs (i.e. the various partial productivity indices). Notice that if the average product of an input is the same in cases 0 and 1, the term for that input equals one and, in that sense, disappears from the total factor productivity index.

In steelmaking, the four principal inputs were labour, capital, fuel, and metallic inputs (mainly pig iron and scrap). 1907 and 1909 are the years chosen for the productivity comparison because they were the years of industrial censuses in the three countries. Unfortunately, as is often the case in historical work, the censuses were not as complete as we would like or indeed presumed in the last section. Output and employment were recorded for the three countries, as was installed horsepower, which shall be used as a measure of the quantity of capital. The consumption of metallic inputs and of fuel, however, was not consistently recorded. Elsewhere[4] I have argued that these inputs were consumed in technologically fixed proportions to output in the early twentieth century. That assumption will be adopted here, in which case, total factor productivity will be measured as

$$\frac{A^1}{A^0} = \left[\frac{Q^1/L^1}{Q^0/L^0}\right]^{.26} \left[\frac{Q^1/K^1}{Q^0/K^0}\right]^{.08} \tag{6}$$

where the shares are as indicated. Labour productivity was 47.5, 70.6 and 84.4 tons per man-year in Britain, Germany and America, while capital productivity (measuring capital by installed horsepower) was 9.0, 14.6 and 7.8 tons per horsepower per year, respectively. Taking the British values as case 0, substitution into equation 6 shows both the German and American industries to have been 15% more efficient than the British (i.e. $A^1/A^0 = 1.15$ for both the German-British and American-British comparisons).

As equation 3 makes clear, the greater efficiency of the German and American industries would tend to give them lower production costs than the British, but that effect might either be attenuated or accentuated by the levels of input prices prevailing in the three countries. We explore that possibility by computing an input price index to estimate the true input price index in equation 3. It is convenient to distinguish four inputs for this calculation - iron ore, fuel, scrap, and labour. The ratios of the prices of these inputs in America to their prices in Britain in 1906-9 were .98, .73, 1.13 and 1.70 respectively. When we use a geometric input price index to aggregate these price relatives we find that, on average, American input prices relative to British were 9% higher (i.e. the index equals 1.09) in 1906-9. Comparing Britain and Germany in the years 1906-13, the relative prices of the inputs were .69, .88, .95 and .72 - all were lower in Germany - and the input price index equals .83.

Equation 3 indicates that production costs in Germany relative to Britain can be computed by multiplying the reciprocal of the German-British total factor productivity index by the German-British input price index. Likewise for America. Table 2 displays the calculations. (Note that the reciprocal of the efficiency index equals $.87 = 1/1.15$.) German costs were 72% of British costs in the first decade of the twentieth century. Germany's greater efficiency and lower input prices made approximately equal contributions to her cost advantage. At the same time American costs

4. Allen, *International Competition*, pp. 919-920.

Table 2: German and American Steelmaking Costs Relative to British

relative cost	=	reciprocal of total factor productivity index	x	input price index
$\frac{w^1 . x^1}{w^0 . x^0}$	=	$\frac{A^0}{A^1}$	.	$\prod_{i=1}^{n} \left[\frac{w_i^1}{w_i^0} \right]^{s_i}$
<u>for German (1) - British (0) Comparison</u>				
.72	=	.87	x	.83
<u>for American (1) - British (0) Comparison</u>				
.95	=	.87	x	1.09

were 95% of British costs. America's costs were lower solely because of her greater efficiency. In fact, American input prices exceeded British prices, mainly because the American steel industry paid wages 70% higher than British wages. To put the matter differently, the superior efficiency of the American industry allowed it to pay higher wages and still produce at lower cost.

IV Conclusion

This paper has summarized recent developments in the theory of production and cost functions, as well as in the theory of index numbers. This theory provides a powerful set of tools to answer questions that have long concerned economic historians. These methods were used to analyze the differences in the cost of producing steel in Germany, Britain and the United States in 1907 and 1909. It was found that the American and German industries were each 15% more efficient than the British. Germany's position in the world market was further enhanced by particularly low input prices, while America's productivity advantage was somewhat offset by the high level of wages prevailing there.

Zusammenfassung:
Neuere Entwicklung in der Produktions- und Kostentheorie sowie in der Indexziffertheorie und ihre Anwendung auf internationale Kosten- und Leistungsunterschiede bei der Stahlherstellung in den Jahren 1907 und 1909

Dieser Beitrag stellt neuere Entwicklungen in der Theorie der Produktions- und Kostenfunktionen sowie der Theorie der Indexziffern zusammenfassend dar. Die Indexziffertheorie bietet das nötige Instrumentarium, um Probleme zu lösen, denen sich Wirtschaftshistoriker schon lange gegenübersahen. Hier wurden diese Methoden angewendet, um die Kostenunterschiede bei der Stahlherstellung in Deutschland, Großbritannien und in den Vereinigten Staaten in den Jahren 1907 und 1909 zu analysieren. Dabei ergab sich, daß sowohl die amerikanische als auch die deutsche Stahlindustrie um 15 Prozent effizienter produzierten als die britische. Darüber hinaus vermochte Deutschland seine Position auf dem Weltmarkt noch durch besonders niedrige Inputpreise zu verbessern, während Amerika seinen Produktivitätsvorteil durch das dort vorherrschende hohe Lohnniveau ziemlich wieder einbüßte.